PUSHOUT ARTIN-SCHELTER REGULAR ALGEBRAS

JUN ZHANG

Abstract. Take k a field and A, B, C k-algebras with $C \subset A, B$ as subalgebra, then there is the pushout $D = A \cup_C B$, which is again a k-algebra. We would like to know when is D an Artin-Schelter regular algebra if A, B, C are. We identify a class of 3 dimensional regular algebras A, B, C where D is regular of dimension 4.

1. Introduction

There is an ongoing effort to classify quantum $\mathbb{P}^3$s, noncommutative projective 3-spaces, or their algebraic correspondence, Artin-Schelter regular algebras of global dimension four. Many families of regular algebras have been discovered recently in [5, 6, 7, 8, 9, 11, 12, 13, 14, 15, 16, 17]. In this paper we construct a new class of Artin-Schelter regular algebras of dimension four as the pushout of two regular algebras of dimension 3.

Take k a field and A, B, C k-algebras with $C \subset A, B$ as subalgebra, then there is the k-algebra pushout $D = A \cup_C B$. In general D is not regular if A, B, C are. For a counter example we have the free algebra $k\langle x, y\rangle = k\langle x\rangle \cup_k k\langle y\rangle$. For simplicity we only work with regular algebras generated in degree 1. As such algebras are well understood up to dimension three [1, 2, 3], the first interesting case is when D has global dimension 4. By the work of [6], such an algebra D is generated by 2, 3, or 4 elements and the projective resolution of the trivial module k_D is given in [6, Proposition 1.4]. When D is generated by 4 elements, then D has 6 quadratic relations, and the projective resolution of k_D is of the form

$$0 \to D(-4) \to D(-3)^{\oplus 4} \to D(-2)^{\oplus 6} \to D(-1)^{\oplus 4} \to D \to k_D \to 0.$$

Suggested by the form of the above resolution, we say such an algebra is *of type (14641)*. In this paper we mainly deal with algebras of type (14641). An algebra of type (14641) is quadratic and Koszul. It is easy to see then both A and B are quadratic regular algebras generated by 3 elements. The algebras A and B are symmetrical, thus we only list the projective resolution of k_A [1]

$$0 \to A(-3) \to A(-2)^{\oplus 3} \to A(-1)^{\oplus 3} \to A \to k_A \to 0$$

Following the notation for D, we say A(and B) is *of type (1331)*. As A and B both have 3 quadratic relations, and D has 6 quadratic relations, the quadratic relations of D must be the same as the quadratic relations of A and B, hence the subalgebra C can not have quadratic relations. A regular algebra with global dimension 2 has a quadratic relation, and a regular algebra with global dimension 3 has either 3 quadratic relations, or 2 cubic relations. Thus C must be a regular algebra of global dimension 3, with 2 cubic relations.

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There are too many dimension 3 algebras to cover at once. In this paper we restrict our algebras A, B, C to the following situation, here all relations are listed in descending order, with $x_4 > x_3 > x_2 > x_1$.

$$A = k\langle x_3, x_1, x_2\rangle/(r_1, r_2, r_3)$$
$$B = k\langle x_1, x_2, x_4\rangle/(r_4, r_5, r_6)$$
$$C = k\langle x_1, x_2\rangle/(r_7, r_8)$$

with the relations r_i defined as follows

$$\begin{aligned} r_1 &= x_3^2 - p_1(x_1, x_2, x_3) \\ r_2 &= x_3x_1 - p_2(x_1, x_2, x_3) \\ r_3 &= x_3x_2 - p_3(x_1, x_2, x_3) \\ r_4 &= x_4x_1 - p_4(x_1, x_2, x_4) \\ r_5 &= x_4x_2 - p_5(x_1, x_2, x_4) \\ r_6 &= x_4^2 - p_6(x_1, x_2, x_4) \\ r_7 &= x_2^2x_1 - p_7(x_1, x_2) \\ r_8 &= x_2x_1^2 - p_8(x_1, x_2) \end{aligned}$$

Our main theorem is the following

Theorem 1.1. *(Theorem 4.10) Under the above restriction on the 3 dimensional regular algebras $C \subset A, B$, any pushout algebra $D = A \cup_C B$ is AS-regular of global dimension 4.*

There are many interesting questions we would like to answer

(a) We would like to continue the computation for all dimension 3 algebras A, B, C. We hypothesize that for any quadratic regular algebra A, B and cubic subalgebra C, their pushout is "generically" a regular algebra of global dimension 4.
(b) We would like to know what kind of algebraic properties is inherited by the pushout algebra. For example, all 3 dimensional regular algebras are Noetherian and we would like to show the dimension 4 pushout algebras are also Noetherian.
(c) We ask the similar question about geometric objects. For example, how the point modules of pushout algebra relate to its subalgebras.
(d) We are interested in higher dimensional extensions.

Here is an outline of the paper: in section 2 we review some basic definitions; in section 3 we give an example that demonstrates many aspect of our computation; in section 4 we define pushout regular algebras and prove their regularity; in section 5 we compute all possible 3 dimensional cubic subalgebras C; in section 6 we compute all possible 3 dimensional quadratic algebra A; in section 7 we give a list of all dimension 4 pushout regular algebras.

2. Definitions

Throughout k is an algebraically closed base field. Everything is over k; in particular, an algebra or a ring is a k-algebra. An algebra D is called *connected graded* if

$$D = k \oplus D_1 \oplus D_2 \oplus \cdots$$

with $1 \in k = D_0$ and $D_i D_j \subset D_{i+j}$ for all i, j. If D is connected graded, then k also denotes the trivial graded module $D/D_{\geq 1}$. In this paper we are working on connected graded algebras. One basic concept we will use is the Artin-Schelter regularity, which we now review. A connected graded algebra D is called *Artin-Schelter regular* or *regular* for short if the following three conditions hold.

(AS1) D has finite global dimension d, and

(AS2) D is *Gorenstein*, namely, there is an integer l such that,

$$\mathrm{Ext}^i_D({}_Dk, D) = \begin{cases} k(l) & \text{if } i = d \\ 0 & \text{if } i \neq d \end{cases}$$

where ${}_Dk$ is the left trivial D-module; and the same condition holds for the right trivial D-module k_D.

(AS3) D has finite Gelfand-Kirillov dimension, i.e., there is a positive number c such that $\dim D_n < c\, n^c$ for all $n \in \mathbb{N}$.

If D is regular, then the global dimension of D is called the*dimension* of D. The notation (l) in (AS2) is the l-th degree shift of graded modules.

In this paper we further assume all graded algebras are generated in degree 1. If D is regular, then by [10, Proposition 3.1.1], the trivial right D-module k_D has a minimal free resolution of the form

$$0 \to P_d \to \cdots P_1 \to P_0 \to k_D \to 0 \tag{E2.0.1}$$

where $P_w = \oplus_{s=1}^{n_w} D(-i_{w,s})$ for some finite integers n_w and $i_{w,s}$. The Gorenstein condition (AS2) implies that the above free resolution is symmetric in the sense that the dual complex of (E2.0.1) is a free resolution of the trivial left D-module after a degree shift. As a consequence, we have $P_0 = D$, $P_d = D(-l)$, $n_w = n_{d-w}$, and $i_{w,s} + i_{d-w,n_w-s+1} = l$ for all w, s.

Regular algebras of dimension three have been classified by Artin, Schelter, Tate and Van den Bergh [1, 2, 3]. A regular algebra of dimension three is generated by either two or three elements. If A is generated by three elements, then A is Koszul and the trivial right A-module k has a minimal free resolution of form

$$0 \to A(-3) \to A(-2)^{\oplus 3} \to A(-1)^{\oplus 3} \to A \to k_A \to 0.$$

If C is generated by two elements, then C is not Koszul and the trivial right C-module k has a minimal free resolution of the form

$$0 \to C(-4) \to C(-3)^{\oplus 2} \to C(-1)^{\oplus 2} \to C \to k_C \to 0.$$

If D is a Noetherian regular algebra of dimension four, then D is generated by 2, 3, or 4 elements [6, Proposition 1.4]. Minimal free resolutions of the trivial module k is listed in [6, Proposition 1.4]. The following lemma is well-known. The transpose of a matrix M is denoted by M^t.

Lemma 2.1. *Let D be a regular graded domain of dimension four. Suppose D is generated by elements x_1, x_2, x_3, x_4 (of degree 1).*

(a) *D is of type (14641), namely, the trivial right D-module k has a free resolution*

$$0 \to D(-4) \xrightarrow{\partial_4} D^{\oplus 4}(-3) \xrightarrow{\partial_3} D^{\oplus 6}(-2) \xrightarrow{\partial_2} D^{\oplus 4}(-1) \xrightarrow{\partial_1} D \xrightarrow{\partial_0} k_D \to 0 \tag{E2.1.1}$$

where $D^{\oplus n}$ is the free right D-module written as an $n \times 1$ matrix.

(b) *∂_0 is the augmentation map with*$\ker \partial_0 = D_{\geq 1}$.

(c) ∂_1 *is given by the left multiplication by* (x_1, x_2, x_3, x_4).
(d) ∂_2 *is the left multiplication by a* 4×6*-matrix* $R = (r_{ij})_{4\otimes 6}$ *such that* $r_i := \sum_{i=1}^{4} x_i r_{ij}$, *for* $i = 1, \dots, 6$, *are the 6 relations of* D.
(e) ∂_3 *is the left multiplication by a* 6×4*-matrix* $T = (t_{ij})_{6\times 4}$.
(f) ∂_4 *is the left multiplication by* $(x'_1, x'_2, x'_3, x'_4)^t$ *where* $\{x'_1, x'_2, x'_3, x'_4\}$ *is a set of generators of* D. (*So each* x'_i *is a* k*-linear combination of* $\{x_i\}_{i=1}^{4}$.)
(g) $(x_1, x_2, x_3, x_4)R = 0$, $RT = 0$, $T(x'_1, x'_2, x'_3, x'_4)^t = 0$.

The dual complex of (E2.1.1) is obtained by applying the functor $(-)^\vee := \mathrm{Hom}_D(-, D)$ to (E2.1.1). Condition (AS2) implies that the dual complex of (E2.1.1) is a free resolution of the left D-module $k(4)$:

$$0 \leftarrow_D k(4) \leftarrow D(4) \xleftarrow{\partial_4^\vee} D^{\oplus 4}(3) \xleftarrow{\partial_3^\vee} D^{\oplus 6}(2) \xleftarrow{\partial_2^\vee} D^{\oplus 4}(1) \xleftarrow{\partial_1^\vee} D \leftarrow 0.$$

Lemma 2.1(f) follows from this observation. Other parts of Lemma 2.1 are clear.

We introduce the following technical condition, referred to as *TC1*, as we use it frequently.

Definition 2.2. Technical Condition *TC1*:
We assume D is a quadratic domain of the form

$$D = k < x_1, x_2, x_3, x_4 > /(r_1, r_2, r_3, r_4, r_5, r_6)$$

where $\{x_1, \dots, x_4\}$ is a set of degree one generators and $\{r_1, \dots, r_6\}$ is a set of quadratic relations. Further assume D has Hilbert Series $H_D(t) = (1-t)^{-4}$ (AS3), and a complex (not necessarily a resolution) of the form

$$0 \to D(-4) \xrightarrow{X^t} D(-3)^{\oplus 4} \xrightarrow{ST} D(-2)^{\oplus 6} \xrightarrow{R} D(-1)^{\oplus 4} \xrightarrow{X} D \to k_D \to 0 \quad (2.3)$$

Where $X = [x_1, x_2, x_3, x_4]$. R is a 4×6 matrix and T is a 6×4 matrix, both with entries in A_1, defined by

$$XR = [r_1, r_2, r_3, r_4, r_5, r_6]$$

$$TX^t = [r_1, r_2, r_3, r_4, r_5, r_6]^t$$

S is a 6×6 invertible matrix with entries in k.
If the algebra D satisfies all of the above conditions, then we say D satisfies *TC1*.
This condition mimics the k resolution from lemma 2.1, with $\partial_3 = ST$ and $\partial_4 = X^t$.

Lemma 2.4. *Assume* D *satisfies TC1. If in addition the partial complex*

$$0 \to D(-4) \xrightarrow{X^t} D(-3)^{\oplus 4} \xrightarrow{T} D(-2)^{\oplus 6} \to \cdots \quad (2.5)$$

is exact, then the complex (2.3) is exact and the algebra D *has global dimension four (AS1).*

Proof. Since S is invertible, the exactness of the complex (2.5) implies the complex (2.3) is exact at the $D(-3)^{\oplus 4}$ and $D(-4)$ positions. That the complex (2.3) is exact at the k_D and D positions is clear. A result of Govorov [4] gives (2.3) exact at the $D(-1)^{\oplus 4}$ position. Lastly $H_D(t) = (1-t)^{-4}$ and exactness at all other positions of (2.3) give us exactness at $D(-2)^{\oplus 6}$ position.
Thus the complex (2.3) is exact and $\mathrm{gldim}(D) = \mathrm{pdim}(k_D) = 4$. □

We can apply lemma 2.4 to the dual complex of 2.3 to get

Lemma 2.6. *Assume D satisfies the conditions of lemma 2.4, hence (AS1) and (AS3), and the following partial complex is exact.*

$$0 \to D^{op}(-4) \xrightarrow{X^t} D^{op}(-3)^{\oplus 4} \xrightarrow{R^t} D^{op}(-2)^{\oplus 6} \to \cdots \tag{2.7}$$

Then the dual complex of 2.3 is also exact and D satisfies (AS2), hence is regular. In particular, if $D^{op} \cong D$ then D is AS-regular.

3. An Example

Before introduce the formal definition of pushout regular algebra, we give a short example to help establish notation, and to help demonstrate the computation techniques we use.

Define algebra A as follows,

$$A = k\langle x_3, x_1, x_2\rangle/(r_1 = x_3^2 - x_1x_2 - x_2x_1, r_2 = x_3x_1 + x_1x_3, r_3 = x_3x_2 + x_2x_3)$$

It is easy to see that A is a quadratic AS-regular algebra of global dimension 3. In the algebra A there are two different ways of expressing $x_3^2x_1$. One way is to first use r_1 to write $(x_3^2)x_1 = (x_1x_2 + x_2x_1)x_1$. The other way is to use r_2 to write $x_3(x_3x_1) = x_3(-x_1x_3)$, then simplify using r_2 and r_1. The process of writing $x_3^2x_1$ as $(x_3^2)x_1$ and $x_3(x_3x_1)$ is called *resolving*. In this case resolving $x_3^2x_1$ shows that

$$\begin{aligned} x_3^2x_1 &= (x_3^2)x_1 = x_2x_1^2 + x_1x_2x_1 \\ &= x_3(x_3x_1) = -x_3x_1x_3 = x_1x_3^2 = x_1x_2x_1 + x_1^2x_2 \end{aligned}$$

Thus we have the cubic relation $x_2x_1^2 - x_1x_2^2$. We resolve $x_3^2x_2$ and x_3^3 similarly. This shows that in A we also have the following two cubic relations

$$\begin{aligned} r_7 &= x_2^2x_1 - x_1x_2^2 \\ r_8 &= x_2x_1^2 - x_1^2x_2 \end{aligned}$$

It is an easy check, using Bergman's Diamond lemma, that the algebra A has no other ambiguities.

We define algebras B and C as

$$B = k\langle x_1, x_2, x_4\rangle/(r_4 = x_4x_1 + x_1x_4, r_5 = x_4x_2 + x_2x_4, r_6 = x_4^2 - x_1x_2 - x_2x_1)$$
$$C = k\langle x_1, x_2\rangle/(r_7 = x_2^2x_1 - x_1x_2^2, r_8 = x_2x_1^2 - x_1^2x_2)$$

In this case $B \cong A$ and C is a dimension 3 regular subalgebra of both A and B.

Define $D = A \cup_C B$, D is the pushout of $C \hookrightarrow A$, and $C \hookrightarrow B$. In terms of generators and relations, we have

$$D = k\langle x_3, x_1, x_2, x_4\rangle/(r_1, r_2, r_3, r_4, r_5, r_6)$$

We give explicit k resolutions for A and D. A minimal free resolution of k_A is

$$0 \to A(-3) \xrightarrow{X_A^t} A(-2)^{\oplus 3} \xrightarrow{R_AS_A = Q_AT_A} A(-1)^{\oplus 3} \xrightarrow{X_A} A \to k_A \to 0 \tag{3.1}$$

Where $X_A = [x_3, x_1, x_2]$, and

$$R_A = \begin{bmatrix} x_3 & x_1 & x_2 \\ -x_2 & x_3 & 0 \\ -x_1 & 0 & x_3 \end{bmatrix}, \quad S_A = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & -1 \\ 0 & -1 & 0 \end{bmatrix},$$

$$T_A = \begin{bmatrix} x_3 & -x_2 & -x_1 \\ x_1 & x_3 & 0 \\ x_2 & 0 & x_3 \end{bmatrix}, \quad Q_A = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & -1 \\ 0 & -1 & 0 \end{bmatrix},$$

$$R_A S_A = Q_A T_A = \begin{bmatrix} x_3 & -x_2 & -x_1 \\ -x_2 & 0 & -x_3 \\ -x_1 & -x_3 & 0 \end{bmatrix}$$

A minimal free resolution of k_D is

$$0 \to D(-4) \xrightarrow{X^t} D(-3)^{\oplus 4} \xrightarrow{ST} D(-2)^{\oplus 6} \xrightarrow{R} D(-1)^{\oplus 4} \xrightarrow{X} D \to k_D \to 0 \tag{3.2}$$

Where $X = [x_3, x_1, x_2, x_4]$, and

$$R = \begin{bmatrix} x_3 & x_1 & x_2 & 0 & 0 & 0 \\ -x_2 & x_3 & 0 & x_4 & 0 & -x_2 \\ -x_1 & 0 & x_3 & 0 & x_4 & -x_1 \\ 0 & 0 & 0 & x_1 & x_2 & x_4 \end{bmatrix}$$

$$T = \begin{bmatrix} x_3 & -x_2 & -x_1 & 0 \\ x_1 & x_3 & 0 & 0 \\ x_2 & 0 & x_3 & 0 \\ 0 & x_4 & 0 & x_1 \\ 0 & 0 & x_4 & x_2 \\ 0 & -x_2 & -x_1 & x_4 \end{bmatrix}$$

$$S = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & -1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & -1 \end{bmatrix}$$

With the matrix $RST = 0$ given as follows,

$$\begin{bmatrix} x_3^2 - x_1x_2 - x_2x_1 & -x_3x_2 - x_2x_3 & -x_1x_3 - x_3x_1 & 0 \\ -x_2x_3 - x_3x_2 & x_2^2 - x_2^2 & x_2x_1 - x_3^2 + x_4^2 - x_2x_1 & x_2x_4 + x_4x_2 \\ -x_1x_3 - x_3x_1 & x_1x_2 - x_3^2 + x_4^2 - x_1x_2 & x_1^2 - x_1^2 & x_1x_4 + x_4x_1 \\ 0 & x_1x_2 + x_2x_1 - x_4^2 & x_4x_1 + x_1x_4 & x_1x_2 + x_2x_1 - x_4^2 \end{bmatrix}$$

The complex 3.2 is exact by theorem 4.9 of next section. It is easy to see that $H_D(t) = (1-t)^{-4}$ and $D^{op} \cong D$, thus by lemma 2.6 the algebra D is regular.

Remark 3.3. (a) In the remainder of the paper we will follow the notations in the above example.

(b) In this example we choose $B \cong A$. In general B do not have to be isomorphic to A.

(c) Although we have the relations of algebra D, the resolution of k_D is one chain longer than the resolution of k_A. It is a non-trivial task to find a 6×6 invertible matrix S as required by *TC1*.

4. Definition and Regularity of Pushout Algebra

We introduce an ordering on the monomials. We order first by degree, from highest to lowest. If two monomials have the same degree, we compare them from left to right according to the ordering $x_4 > x_3 > x_2 > x_1$. By Bergman's diamond lemma, we can reduce any element to a sum of basis monomials with minimal order. The basis we thus obtain is similar to Gröbner basis for commutative algebras. Our ordering extend naturally to a finite set of relations. From now on by a set of relations we mean a set of relations with minimal order.

Definition 4.1. Pushout Algebra:
Take A, B, C, each a regular algebra of global dimension 3, given as follows,

$$A = k\langle x_3, x_1, x_2\rangle/(r_1, r_2, r_3)$$
$$B = k\langle x_1, x_2, x_4\rangle/(r_4, r_5, r_6)$$
$$C = k\langle x_1, x_2\rangle/(r_7, r_8)$$

where the set of relations $\{r_i\}$, in minimal ordering, is given as follows,

$$\begin{aligned}
r_1 &= x_3^2 - p_1(x_1, x_2, x_3)\\
r_2 &= x_3x_1 - p_2(x_1, x_2, x_3)\\
r_3 &= x_3x_2 - p_3(x_1, x_2, x_3)\\
r_4 &= x_4x_1 - p_4(x_1, x_2, x_4)\\
r_5 &= x_4x_2 - p_5(x_1, x_2, x_4)\\
r_6 &= x_4^2 - p_6(x_1, x_2, x_4)\\
r_7 &= x_2^2x_1 - p_7(x_1, x_2)\\
r_8 &= x_2x_1^2 - p_8(x_1, x_2)
\end{aligned}$$

We further require that $C = A \cap B$ as subalgebra of A and B. All ambiguities in C resolve, and after reduction by r_7 and r_8, all ambiguities in A and B resolve.

Under the above assumptions, we can form the k-algebra pushout $D = A \cup_C B$. From now on, we use the term *Pushout Algebra* for the above situation only.

Remark 4.2. As we are interested in D quadratic algebra only. It is clear that A, B both have to be quadratic and C must be cubic. The additional requirement on the relations is to break the computation into more manageable pieces. For instance we have the following quick computation of Hilbert Series.

Lemma 4.3. *The algebra D has Hilbert Series $H_D(t) = (1-t)^{-4}$ and basis elements*

$$x_1^p(x_2x_1)^l x_2^m x_3^{0,1}(x_4x_3)^n x_4^{0,1}$$

Proof. Algebra D has the same Hilbert Series and basis as the following algebra

$$k\langle x_3, x_1, x_2, x_4\rangle/(x_4^2, x_4x_2, x_4x_1, x_3^2, x_3x_2, x_3x_1, x_2^2x_1, x_2x_1^2)$$

□

Define $X_A = [x_3, x_1, x_2]$, $X_B = [x_1, x_2, x_4]$, and T_A, T_B 3×3 matrices with $T_AX_A^t = [r_1, r_2, r_3]^t$, $T_BX_B^t = [r_4, r_5, r_6]^t$. Since A and B are regular, we have k resolutions

$$0 \to A(-3) \xrightarrow{X_A^t} A(-2)^{\oplus 3} \xrightarrow{Q_AT_A} A(-1)^{\oplus 3} \xrightarrow{X_A} A \to k_A \to 0 \tag{4.4}$$

$$0 \to B(-3) \xrightarrow{X_B^t} B(-2)^{\oplus 3} \xrightarrow{Q_BT_B} B(-1)^{\oplus 3} \xrightarrow{X_B} B \to k_B \to 0 \tag{4.5}$$

Where Q_A, Q_B are 3×3 non-singular scalar matrices.

For the following results we assume our pushout algebra D satisfies *TC1*, namely there is a complex of right D modules

$$0 \to D(-4) \xrightarrow{X^t} D(-3)^{\oplus 4} \xrightarrow{ST} D(-2)^{\oplus 6} \xrightarrow{R} D(-1)^{\oplus 4} \xrightarrow{X} D \to k_D \to 0 \tag{4.6}$$

where $X = [x_3, x_1, x_2, x_4]$, R is a 4×6 matrix with $XR = [r_1, \ldots, r_6]$, T is a 6×4 matrix with $TX^t = [r_1, \ldots, r_6]^t$ and S a 6×6 scalar matrix.

We will show that the complex 4.6 is exact at the $D(-3)$ position. Notice the upper left and lower right 3×3 blocks of the matrix T is respectively the matrices T_A and T_B, with the remaining entries 0. From the exact sequences 4.4 and 4.5 we have $\mathrm{im}(X_A^t)=\ker(T_A)$ and $\mathrm{im}(X_B^t)=\ker(T_B)$. View the algebra A as subalgebra of D, we have the following,

Lemma 4.7. *Take $[d_1,d_2,d_3]^t\in D^{\oplus 3}$ with $T_A[d_1,d_2,d_3]^t=0$, then*

$$[d_1,d_2,d_3]^t\in[x_3,x_1,x_2]^tD$$

Proof. In lemma 4.3, we have a basis element of D to have the form

$$x_1^p(x_2x_1)^lx_2^mx_3^{0,1}(x_4x_3)^nx_4^{0,1}$$

Define $g_{2n}=(x_4x_3)^n$ and $g_{2n+1}=(x_4x_3)^nx_4$. Then an element $d\in D$ can be uniquely written as

$$d=\sum a_ng_n$$

with each $a_n\in A$ a sum of basis elements in A. Take another element $a'\in A$, then

$$a'd=a'(\sum a_ng_n)=\sum(a'a_n)g_n=\sum a'_ng_n$$

Here $a'_n=a'a_n\in A$ is also a sum of basis elements in A, and a'_ng_n is a sum of basis elements in D. Follow this notation, we write

$$[d_1,d_2,d_3]^t=[\sum a_{1n}g_n,\sum a_{2n}g_n,\sum a_{3n}g_n]^t$$

Let $[t_1,t_2,t_3]\in A^{\oplus 3}$ be a row in T_A, we have

$$\begin{aligned}0 &= [t_1,t_2,t_3][d_1,d_2,d_3]^t\\ &= [t_1,t_2,t_3][\sum a_{1n}g_n,\sum a_{2n}g_n,\sum a_{3n}g_n]^t\\ &= \sum t_1a_{1n}g_n+\sum t_2a_{2n}g_n+\sum t_3a_{3n}g_n\\ &= \sum(t_1a_{1n}+t_2a_{2n}+t_3a_{3n})g_n\end{aligned}$$

From the above we conclude that for each n, $0=t_1a_{1n}+t_2a_{2n}+t_3a_{3n}$. Since we have $\mathrm{im}(X_A^t)=\ker(T_A)$, we have for each n,

$$[a_{1n},a_{2n},a_{3n}]^t\in[x_3,x_1,x_2]^tA$$

This shows that

$$[d_1,d_2,d_3]^t=\sum[a_{1n},a_{2n},a_{3n}]^tg_n\in[x_3,x_1,x_2]^tD$$

$\square$

Apply the same argument to $B\subset D$, we have

Lemma 4.8. *Take $[d_1,d_2,d_3]^t\in D^{\oplus 3}$ with $T_B[d_1,d_2,d_3]^t=0$, then*

$$[d_1,d_2,d_3]^t\in[x_1,x_2,x_4]^tD$$

Theorem 4.9. *If the pushout algebra D satisfies TC1, then* $\mathrm{gldim}(D)=4$.

Proof. By lemma 4.3, D has Hilbert Series $H_D(t)=(1-t)^{-4}$.
From our form of basis we can easily see that left multiplication by x_1 is injective, hence left multiplication by X^t is also injective. By lemma 2.4, to show $\mathrm{gldim}(D)=$

4, we only need to show $\ker(T_D) \subset \operatorname{im}(X^t)$.
Define matrices T_1, T_2 as follows,

$$T_1 = \begin{bmatrix} & & & 0 \\ & T_A & & 0 \\ & & & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}, \quad T_2 = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & & & \\ 0 & & T_B & \\ 0 & & & \end{bmatrix}$$

By lemma 4.7, 4.8, we have

$$\ker(T_1) = \begin{bmatrix} x_3 d_1 \\ x_1 d_1 \\ x_2 d_1 \\ d_2 \end{bmatrix}$$

$$\ker(T_2) = \begin{bmatrix} d_4 \\ x_1 d_3 \\ x_2 d_3 \\ x_4 d_3 \end{bmatrix}$$

From the above we have

$$\ker(T_D) \subset \ker(T_1) \cap \ker(T_2) = \begin{bmatrix} x_3 d_1 \\ x_1 d_1 \\ x_2 d_1 \\ d_2 \end{bmatrix} \cap \begin{bmatrix} d_4 \\ x_1 d_3 \\ x_2 d_3 \\ x_4 d_3 \end{bmatrix} = \operatorname{im}(\begin{bmatrix} x_3 \\ x_1 \\ x_2 \\ x_4 \end{bmatrix})$$

□

By lemma 7.3 of section 7 we have that all possible pushout algebra D's satisfy *TC1*. By lemma 6.2 of section 6 we have for all pushout algebra D, $D^{op} \cong D$. Thus combining theorem 4.9 and lemma 2.6 we have our main result

Theorem 4.10. *All pushout algebras, defined as in 4.1, are regular of global dimension 4.*

5. Classification of Algebra C

In this section we identify all possible dimension 3 regular cubic algebra C with relations

$$C = k\langle x_1, x_2\rangle/(r_7, r_8)$$

where the relations r_7 and r_8 are

$$\begin{aligned} r_7 &= x_2^2 x_1 - c_1 x_2 x_1 x_2 - c_2 x_1 x_2^2 - c_3 x_1 x_2 x_1 - c_4 x_1^2 x_2 - c_5 x_1^3 \\ r_8 &= x_2 x_1^2 - c_6 x_1 x_2^2 - c_7 x_1 x_2 x_1 - c_8 x_1^2 x_2 - c_9 x_1^3 \end{aligned}$$

with coefficients $c_i \in k$.

Resolving the ambiguity $x_2^2x_1^2$ in C gives us the equation

$$\begin{aligned}
0 &= (x_2^2x_1)x_1 - x_2(x_2x_1^2) \\
&= -c_6x_2x_1x_2^2 + (c_1 - c_7)x_2x_1x_2x_1 - c_6c_8x_1x_2^3 \\
&\quad +(c_1c_2 - c_1c_6c_9 - c_7c_8)x_1x_2x_1x_2 + (c_2^2 + c_3c_6 - c_8^2 - c_6c_7c_9 - c_2c_6c_9)x_1^2x_2^2 \\
&\quad +(c_2c_3 + c_4 + c_3c_7 - c_3c_6c_9 - c_8c_9 - c_7^2c_9)x_1^2x_2x_1 \\
&\quad +(c_2c_4 + c_3c_8 - c_4c_6c_9 - c_8c_9 - c_7c_8c_9)x_1^3x_2 \\
&\quad +(c_5 + c_2c_5 + c_3c_9 - c_5c_6c_9 - c_9^2 - c_7c_9^2)x_1^4
\end{aligned}$$

Solving for the coefficients c_i's in the above equation gives the following set of equations,

$$\begin{aligned}
c_6 &= 0 \\
c_7 &= c_1 \\
0 &= c_1(c_2 - c_8) \\
0 &= (c_2 - c_8)(c_2 + c_8) \\
0 &= (c_1 + c_2)c_3 + c_4 - (c_1^2 + c_8)c_9 \\
0 &= c_2c_4 + c_3c_8 - (1 + c_1)c_8c_9 \\
0 &= (c_2 + 1)c_5 + c_3c_9 - (1 + c_1)c_9^2
\end{aligned}$$

Some of the solutions to the above system give algebras that are not domain, hence not regular. We have the following lemma for the other solutions

Lemma 5.1. *Any domain satisfying the above system of equations are regular.*

Proof. To verify a solution is a domain is computationally intensive. Fortunately we only have to exclude the algebras that are obviously not domain. It is easy to see the remaining solutions have Hilbert Series equal to $(1-t)^{-2}(1-t^2)^{-1}$, hence satisfy (AS3). We verify (AS1) by constructing a resolution $P^\cdot \to k_C$ for each algebra. We omit the resolutions here as they play no future role. Finally for each solution C, we have $C^{op} \cong C$, and the dual resolution of $P^\cdot \to k_C$ gives us (AS2). $\square$

We list the relations of regular algebra C's below. It is worth noting that some of the algebras are only determined up to (linear) isomorphisms. We try to choose the basis that give us shorter relations but our choices are by no means optimal.

C0 This is a special case of the next algebra, C1.

$$\begin{aligned}
r_7 &= x_2^2x_1 - x_1x_2^2 \\
r_8 &= x_2x_1^2 - x_1^2x_2
\end{aligned}$$

C1

$$c_2 \neq 0$$

$$\begin{aligned}
r_7 &= x_2^2x_1 - c_1x_2x_1x_2 - c_2x_1x_2^2 \\
r_8 &= x_2x_1^2 - c_1x_1x_2x_1 - c_2x_1^2x_2
\end{aligned}$$

C2

$$c_2 \neq 0$$

$$\begin{aligned}
r_7 &= x_2^2x_1 - c_2x_1x_2^2 \\
r_8 &= x_2x_1^2 + c_2x_1^2x_2
\end{aligned}$$

C3

$$\begin{aligned} r_7 &= x_2^2x_1 + x_1x_2^2 - x_1^3 \\ r_8 &= x_2x_1^2 - x_1^2x_2 \end{aligned}$$

C4

$$c_1 \neq 2$$

$$\begin{aligned} r_7 &= x_2^2x_1 - c_1x_2x_1x_2 + x_1x_2^2 - c_5x_1^3 \\ r_8 &= x_2x_1^2 - c_1x_1x_2x_1 + x_1^2x_2 \end{aligned}$$

C5

$$c_9 \neq 0$$

$$\begin{aligned} r_7 &= x_2^2x_1 - 2x_2x_1x_2 + x_1x_2^2 - 3c_9x_1x_2x_1 \\ r_8 &= x_2x_1^2 - 2x_1x_2x_1 + x_1^2x_2 - c_9x_1^3 \end{aligned}$$

C6

$$c_3 \neq 0$$

$$\begin{aligned} r_7 &= x_2^2x_1 - 2x_2x_1x_2 + x_1x_2^2 - c_3x_1x_2x_1 + c_3x_1^2x_2 - c_5x_1^3 \\ r_8 &= x_2x_1^2 - 2x_1x_2x_1 + x_1^2x_2 \end{aligned}$$

C7 One of c_4, c_9 is non-zero.

$$\begin{aligned} r_7 &= x_2^2x_1 - x_1x_2^2 - (c_9 - c_4)x_1x_2x_1 - c_4x_1^2x_2 - \frac{1}{2}c_4c_9x_1^3 \\ r_8 &= x_2x_1^2 - x_1^2x_2 - c_9x_1^3 \end{aligned}$$

6. Classification of Algebra A

In this section we identify all possible regular quadratic algebras A over each choice of algebra C. By symmetry the algebra B's have the same classification. We list the relations r_i of A below

$$A = k\langle x_3, x_1, x_2\rangle/(r_1, r_2, r_3)$$

with relations r_1, r_2, r_3 given as

$$\begin{aligned} r_1 &= x_3^2 - a_6x_2x_3 - a_5x_2^2 - a_4x_2x_1 - a_3x_1x_3 - a_2x_1x_2 - a_1x_1^2 \\ r_2 &= x_3x_1 - a_{26}x_2x_3 - a_{25}x_2^2 - a_{24}x_2x_1 - a_{23}x_1x_3 - a_{22}x_1x_2 - a_{21}x_1^2 \\ r_3 &= x_3x_2 - a_{16}x_2x_3 - a_{15}x_2^2 - a_{14}x_2x_1 - a_{13}x_1x_3 - a_{12}x_1x_2 - a_{11}x_1^2 \end{aligned}$$

We reduce the relations by linear change of variable. Any change of variables must preserve the relations of C inside A, thus we can scale x_2, or change x_3 to $x_3 + \alpha x_1 + \beta x_2$. If $a_{26} = 0$ then we can choose $a_{24} = a_{12} = 0$ after the change of variable $x_3 \to x_3 + (a_{13}a_{24} + a_{12})x_1 + a_{24}x_2$. Otherwise $a_{26} \neq 0$ and first we can choose $a_{26} = 1$ by scaling x_2, then we can choose $a_{24} = a_{25} = 0$ after the change of variable $x_3 \to x_3 - (a_{24} + a_{25})x_1 - a_{25}x_2$. To sum it up, we have, in general

$$\begin{aligned} r_1 &= x_3^2 - a_6x_2x_3 - a_5x_2^2 - a_4x_2x_1 - a_3x_1x_3 - a_2x_1x_2 - a_1x_1^2 \\ r_2 &= x_3x_1 - a_{26}x_2x_3 - a_{25}x_2^2 - a_{23}x_1x_3 - a_{22}x_1x_2 - a_{21}x_1^2 \\ r_3 &= x_3x_2 - a_{16}x_2x_3 - a_{15}x_2^2 - a_{14}x_2x_1 - a_{13}x_1x_3 - a_{12}x_1x_2 - a_{11}x_1^2 \end{aligned}$$

with either $a_{26} = a_{12} = 0$, or $a_{25} = 0, a_{26} = 1$.

In the special case of algebra $C0$, in addition to the previous change of variable we can also change x_2 to $x_2+\gamma x_1$. In this case we can reduce r_1, r_2, r_3 further to have

$$\begin{aligned}
r_1 &= x_3^2 - a_6x_2x_3 - a_5x_2^2 - a_4x_2x_1 - a_3x_1x_3 - a_2x_1x_2 - a_1x_1^2\\
r_2 &= x_3x_1 - a_{26}x_2x_3 - a_{25}x_2^2 - a_{23}x_1x_3 - a_{22}x_1x_2 - a_{21}x_1^2\\
r_3 &= x_3x_2 - a_{16}x_2x_3 - a_{15}x_2^2 - a_{14}x_2x_1 - a_{13}x_1x_3 - a_{12}x_1x_2 - a_{11}x_1^2\\
r_7 &= x_2^2x_1 - x_1x_2^2\\
r_8 &= x_2x_1^2 - x_1^2x_2
\end{aligned}$$

with either $a_{26}=1, a_{25}=a_{23}=0$ or $a_{13}=a_{26}=a_{12}=0$.

Remark 6.1. We use $Ai.j$ to denote the algebra of type j, lying over Ci.

Only algebras $C0$, $C1$, and $C6$ have possible regular algebra A containing them. We have the following lemma

Lemma 6.2. *The regular algebras* $A6.1$*,* $A1.1$*,* $A1.2$*,* $A1.3$*,* $A0.1$*,* $A0.2$*,* $A0.3$*,* $A0.4$ *are the only possible algebras containing any algebra* C*. In addition, for each algebra* A*, we have* $A^{op}\cong A$ *by the change of variable* x_1 *to* $-x_1$*. Thus for any pushout algebra* $D=A\cup_C B$ *we also have* $D^{op}\cong D$*, by the same change of variable.*

Proof. Here we give an outline, while omitting the details. In principle the computations can be carried out explicitly. We use the package "Affine" of the program "Maxima" for most of the Gröbner basis computation. This leads to several systems of equations which we then use "Maple" to solve. First we solve the system of equations come from resolving the ambiguities $x_3^3, x_3^2x_1, x_3^2x_2, x_3x_2^2x_1, x_3x_2x_1^2, x_2^2x_1^2$. Then we check if the solutions generates the cubic relations r_7 and r_8. Once this is done we then find a linear resolution for k_A for each algebra. This is equivalent to find a non-singular 3×3 scalar matrix S_A with $R_AS_AX_A^t=0$. We then notice from the list below that for each solution we have $A^{op}\cong A$. □

A6.1

$$p^2=1$$

$$\begin{aligned}
r_1 &= x_3^2 - x_2x_1 + x_1x_2 + a_1x_1^2\\
r_2 &= x_3x_1 - px_1x_3\\
r_3 &= x_3x_2 - px_2x_3 + p(1-a_1)x_1x_3\\
r_7 &= x_2^2x_1 - 2x_2x_1x_2 + x_1x_2^2 - 2x_1x_2x_1 + 2x_1^2x_2 + 2a_1(1-a_1)x_1^3\\
r_8 &= x_2x_1^2 - 2x_1x_2x_1 + x_1^2x_2
\end{aligned}$$

A1.1

$$a_{23}\neq 0,\ c_2\neq 0$$

$$\begin{aligned}
r_1 &= x_3^2 - x_2x_1 - c_2a_{23}^{-2}x_1x_2\\
r_2 &= x_3x_1 - a_{23}x_1x_3\\
r_3 &= x_3x_2 - a_{23}^{-1}x_2x_3\\
r_7 &= x_2^2x_1 - (a_{23}^2 - c_2a_{23}^{-2})x_2x_1x_2 - c_2x_1x_2^2\\
r_8 &= x_2x_1^2 - (a_{23}^2 - c_2a_{23}^{-2})x_1x_2x_1 - c_2x_1^2x_2
\end{aligned}$$

A1.2

$$p^2 = -1$$

$$\begin{aligned} r_1 &= x_3^2 - x_2^2 \\ r_2 &= x_3x_1 - px_1x_3 \\ r_3 &= x_3x_2 - x_2x_3 - x_1^2 \\ r_7 &= x_2^2x_1 + x_1x_2^2 \\ r_8 &= x_2x_1^2 + x_1^2x_2 \end{aligned}$$

A1.3

$$p^2 - p + 1 = 0,\ a_{11}a_{25} \neq 1 - p$$

$$\begin{aligned} r_1 &= x_3^2 - x_2x_1 + px_1x_2 \\ r_2 &= x_3x_1 - a_{25}x_2^2 - (1-p)x_1x_3 \\ r_3 &= x_3x_2 - px_2x_3 - a_{11}x_1^2 \\ r_7 &= x_2^2x_1 - (p-1)x_1x_2^2 \\ r_8 &= x_2x_1^2 - (p-1)x_1^2x_2 \end{aligned}$$

A0.1 This is the special case of A1.1 with $a_{23} = p$ and $c_2 = 1$.

$$p^2 = -1$$

$$\begin{aligned} r_1 &= x_3^2 - x_2x_1 + x_1x_2 \\ r_2 &= x_3x_1 - px_1x_3 \\ r_3 &= x_3x_2 + px_2x_3 \\ r_7 &= x_2^2x_1 - x_1x_2^2 \\ r_8 &= x_2x_1^2 - x_1^2x_2 \end{aligned}$$

A0.2

$$\begin{aligned} r_1 &= x_3^2 - x_2^2 \\ r_2 &= x_3x_1 - x_1x_3 \\ r_3 &= x_3x_2 + x_2x_3 - a_{15}x_2^2 - x_1^2 \\ r_7 &= x_2^2x_1 - x_1x_2^2 \\ r_8 &= x_2x_1^2 - x_1^2x_2 \end{aligned}$$

A0.3

$$a_{11}(a_{15} - a_{11}) \neq 1$$

$$\begin{aligned} r_1 &= x_3^2 - x_2^2 - x_1^2 \\ r_2 &= x_3x_1 - x_1x_3 \\ r_3 &= x_3x_2 + x_2x_3 - a_{15}x_2^2 - a_{11}x_1^2 \\ r_7 &= x_2^2x_1 - x_1x_2^2 \\ r_8 &= x_2x_1^2 - x_1^2x_2 \end{aligned}$$

A0.4 Generically regular, with the coefficients a_i satisfying lemma 6.3,

$$\begin{aligned}
r_1 &= x_3^2 - a_5x_2^2 - a_4x_2x_1 - a_4x_1x_2 - a_1x_1^2 \\
r_2 &= x_3x_1 - a_{25}x_2^2 + x_1x_3 - a_{21}x_1^2 \\
r_3 &= x_3x_2 + x_2x_3 - a_{15}x_2^2 - a_{11}x_1^2 \\
r_7 &= x_2^2x_1 - x_1x_2^2 \\
r_8 &= x_2x_1^2 - x_1^2x_2
\end{aligned}$$

The algebra A from section 3 is of this type.

For the algebra $A0.4$, if we define

$$\begin{aligned}
k_1 &= -a_4a_{15} - a_1a_{25} \\
k_2 &= a_4a_{21} + a_5a_{11} \\
k_3 &= a_5 - a_{21}a_{25} \\
k_4 &= a_4 + a_{11}a_{25} \\
k_5 &= a_{11}a_{15} - a_1
\end{aligned}$$

then resolving the ambiguities x_3^3, $x_3^2x_1$, $x_3^2x_2$ give us the following cubic relations,

$$\begin{aligned}
k_1r_7 + k_2r_8 &= 0, \\
k_3r_7 + k_4r_8 &= 0, \\
-k_4r_7 + k_5r_8 &= 0
\end{aligned}$$

Thus for r_1, r_2, r_3 to generate r_7, r_8, we need the matrix

$$\begin{bmatrix} k_1 & k_2 \\ k_3 & k_4 \\ -k_4 & k_5 \end{bmatrix}$$

to have rank 2. This is same as requiring one of the 2×2 minors to be non-singular. Thus one of $k_1k_4 - k_2k_3$ and $k_3k_5 + k_4^2$ must be non-zero.

In addition to generating r_7 and r_8, we need to solve for a 3×3 non-singular scalar matrix S_A with $R_AS_AX_A^t = 0$. Our solution has the form

$$S_A = \begin{bmatrix} s_1 & s_2 & s_3 \\ s_2 & -(a_{11}s_3 + a_{21}s_2 + a_1s_1) & -a_4s_1 \\ s_3 & -a_4s_1 & -(a_{15}s_3 + a_{25}s_2 + a_5s_1) \end{bmatrix}$$

subject to the constraints

$$\begin{aligned}
0 &= -k_4s_3 + k_3s_2 + (-a_{15}k_4 + a_{25}k_5)s_1 \\
0 &= k_5s_3 + k_4s_2 + (a_{11}k_3 + a_{21}k_4)s_1
\end{aligned}$$

A solution for a non-singular S_A is the same as a non-trivial solutions for s_1, s_2, s_3, subjected to the above two equations, and $\det(S_A) \neq 0$, and one of $k_1k_4 - k_2k_3$ or $k_3k_5 + k_4^2$ non-zero. This is true for generic choices of the coefficients a_i. We sum up the above argument in the following lemma,

Lemma 6.3. *The algebra $A0.4$ is regular if there is a non-trivial set of solutions s_1, s_2, s_3 satisfying the following set of conditions,*

$$\begin{aligned} 0 &= -k_4 s_3 + k_3 s_2 + (-a_{15}k_4 + a_{25}k_5)s_1 \\ 0 &= k_5 s_3 + k_4 s_2 + (a_{11}k_3 + a_{21}k_4)s_1 \\ 0 &\neq \det(S_A) \\ 0 &\neq k_1k_4 - k_2k_3 \quad \text{or} \quad 0 \neq k_3k_5 + k_4^2 \end{aligned}$$

Where $k_1 = -a_4a_{15} - a_1a_{25}$, $k_2 = a_4a_{21} + a_5a_{11}$, $k_3 = a_5 - a_{21}a_{25}$, $k_4 = a_4 + a_{11}a_{25}$, $k_5 = a_{11}a_{15} - a_1$, and

$$S_A = \begin{bmatrix} s_1 & s_2 & s_3 \\ s_2 & -(a_{11}s_3 + a_{21}s_2 + a_1s_1) & -a_4s_1 \\ s_3 & -a_4s_1 & -(a_{15}s_3 + a_{25}s_2 + a_5s_1) \end{bmatrix}$$

Thus the algebra $A0.4$ is regular for generic choices of coefficients a_i.

Proof. There are three variables s_1, s_2, s_3 with two equations

$$\begin{aligned} 0 &= -k_4 s_3 + k_3 s_2 + (-a_{15}k_4 + a_{25}k_5)s_1 \\ 0 &= k_5 s_3 + k_4 s_2 + (a_{11}k_3 + a_{21}k_4)s_1 \end{aligned}$$

and two open conditions

$$\begin{aligned} 0 &\neq \det(S_A) \\ 0 &\neq k_1k_4 - k_2k_3 \quad \text{or} \quad 0 \neq k_3k_5 + k_4^2 \end{aligned}$$

Hence there exist non-trivial solutions to s_1, s_2, s_3 for an open dense set of coefficients a_i's. Thus the algebra $A0.4$ is regular for generic a_i's. □

7. Classification of Pushout Algebras

From the list of algebra A's, we can write down the relations for each possible pushout algebra D. It remains to check the condition *TC1*, that is, for each algebra D, solve for a non-singular 6×6 matrix $S = [s_{ij}],\ i, j \in \{1 \dots 6\}$ such that $RST = 0$. Here the matrices R, S, T is as defined in 2.2. This is done through direct computation, together with the help of the following lemma.

Lemma 7.1. *If D is a possible pushout algebra, then its 6×6 matrix S from TC1 has the block form*

$$S = \begin{bmatrix} S_1 & 0 \\ 0 & S_2 \end{bmatrix}$$

where S_1 and S_2 are 3×3 non-singular matrices.

Proof. For this lemma we only need to use the leading term of each relations. Write

$$D = k\langle x_3, x_1, x_2, x_4\rangle/(x_3^2 - f_1, x_3x_1 - f_2, x_3x_2 - f_3, x_4x_1 - f_4, x_4x_2 - f_5, x_4^2 - f_6)$$

We expand the entry $RST_{12} = \sum R_{1i}s_{ij}T_{j2}$ in lowest Gröbner basis. The only x_3x_4 term in it is $s_{14}x_3x_4$. Since $RST = 0$ and there are no relation with x_3x_4, we must have $s_{14} = 0$. Again in RST_{12} we obtain an x_1x_4 term only if $i = 2, j = 4$. The monomials whose reduction give possible x_1x_4 terms are x_4x_1, x_4x_2, x_4^2. These do not appear in RST_{12}. Thus the x_1x_4 term in RST_{12} is $s_{24}x_1x_4$, so we conclude $s_{24} = 0$. Similarly, the x_2x_4 term in RST_{12} is $s_{34}x_2x_4$, so we conclude $s_{34} = 0$.
If we carry out the same analysis for x_3x_4, x_1x_4, x_2x_4 terms in RST_{13}, RST_{14} we get $s_{15} = s_{25} = s_{35} = s_{16} = s_{26} = s_{36} = 0$. Thus the upper right 3×3 block of

S are 0's. Similarly, by computing x_1x_3, x_2x_3, x_4x_3 terms in $RST_{41}, RST_{42}, RST_{43}$ we have the lower left 3×3 block of S are 0's.

□

Remark 7.2. The matrices S_1, S_2 do not have to be the same as the 3×3 matrices S_A, S_B from the algebras A and B. We notice from our computation that if we choose $\det(S_1) = 1$, then we always have $\det(S_2) = -1$. We do not yet know the reason behind it.

The following lemma is proved by solving for a matrix S that satisfies $RST = 0$ for each algebra. We state it without showing the details of computation.

Lemma 7.3. *All possible pushout algebras satisfy condition TC1.*

Remark 7.4. We use $Di.jk$ to denote the algebra $Ai.j \cup_{Ci} Bi.k$. Here $Bi.k$ is isomorphic to $Ai.k$, with x_3 replaced by x_4 and any coefficient a_n replaced by b_n. As there is a large number of pushout algebras, we do not list the relations for most of them and list only any additional constraints arising from solving for condition *TC1*.

Algebras without $A0.4$.

For the algebra $D6.11$ we have $a_1 = b_1$. There is no additional constraint for the algebras $D1.11$, $D1.22$, $D1.33$. For the algebra $D1.12$ we have $c_2 = -1, a_{23}^4 = 1$. For the algebra $D1.13$ we have $c_2 = p-1, a_{23}^4 = p-1$. It is not possible to form $A1.2 \cup B1.3$ as they require incompatible coefficients in the algebra $C1$. There is no additional constraints for the algebras $D0.11$, $D0.12$, $D0.13$, $D0.22$, $D0.23$, $D0.33$. All of the above constraints are exactly what needed to match coefficients in the algebra C's.

Algebras $D0.41, D0.42, D0.43$.

For the algebras $D0.41$, $D0.42$, $D0.43$, $D0.44$ the solutions of the 6×6 matrix S has the form

$$S = \begin{bmatrix} S_A & 0 \\ 0 & S_2 \end{bmatrix}$$

That is, the upper left 3×3 block of S is the same as the 3×3 matrix S_A from the algebra $A0.4$. Thus we can keep using the notations introduced in lemma 6.3. Let k_i's and s_i's as defined in lemma 6.3. Let also

$$\begin{aligned} s_4 \quad = \quad & -a_{11}k_3s_3 + (a_{25}k_5 - a_{15}k_4 - a_{21}k_3)s_2 \\ & +(k_3k_5 + k_4^2 - a_{11}a_{25}k_4 - a_{11}a_{15}k_3)s_1 \end{aligned}$$

Then the algebras $D0.41$, $D0.42$, $D0.43$ are regular if the coefficients a_i satisfy lemma 6.3 and the extra open condition $s_4 \neq 0$. It is easy to see that this is again true for generic choices of a_i's.

Algebra $D0.44$.

Finally we list the relations of algebra $D0.44 = A0.4 \cup_{C0} B0.4$ as follows,

$$\begin{aligned}
r_1 &= x_3^2 - a_5 x_2^2 - a_4 x_2 x_1 - a_4 x_1 x_2 - a_1 x_1^2 \\
r_2 &= x_3 x_1 - a_{25} x_2^2 + x_1 x_3 - a_{21} x_1^2 \\
r_3 &= x_3 x_2 + x_2 x_3 - a_{15} x_2^2 - a_{11} x_1^2 \\
r_4 &= x_4 x_1 - b_{25} x_2^2 + x_1 x_4 - b_{21} x_1^2 \\
r_5 &= x_4 x_2 + x_2 x_4 - b_{15} x_2^2 - b_{11} x_1^2 \\
r_6 &= x_4^2 - b_5 x_2^2 - b_4 x_2 x_1 - b_4 x_1 x_2 - b_1 x_1^2
\end{aligned}$$

Since $B0.4 \cong A0.4$, after identifying a_i's with b_i's, the coefficients b_i's also satisfy lemma 6.3. For the algebra $B0.4$ we define l_i, S_B, t_1, t_2, t_3, t_4 from the coefficients b_i's similarly as we defined k_i, S_A, s_1, s_2, s_3, s_4 in lemma 6.3 from the coefficients a_i's. The algebra $D0.44$ is regular if there is a non-trivial set of solutions $\{s_1, s_2, s_3, t_1, t_2, t_3\}$ satisfying

$$\begin{aligned}
0 &= -k_4 s_3 + k_3 s_2 + (-a_{15} k_4 + a_{25} k_5) s_1 \\
0 &= k_5 s_3 + k_4 s_2 + (a_{11} k_3 + a_{21} k_4) s_1 \\
0 &\neq \det(S_A) \\
0 &\neq k_1 k_4 - k_2 k_3 \quad \mathsf{or} \quad 0 \neq k_3 k_5 + k_4^2 \\
0 &= -l_4 t_3 + l_3 t_2 + (-b_{15} l_4 + b_{25} l_5) t_1 \\
0 &= l_5 t_3 + l_4 t_2 + (b_{11} l_3 + b_{21} l_4) t_1 \\
0 &\neq \det(S_B) \\
0 &\neq l_1 l_4 - l_2 l_3 \quad \mathsf{or} \quad 0 \neq l_3 l_5 + l_4^2 \\
0 &= s_4 + t_4
\end{aligned}$$

There are six variables, five equations, with extra open conditions, hence we have non-trivial solutions for an open set of a_i's and b_j's. We conclude the algebra $D0.44$ is generically regular. The algebra D from section 3 is of type $D0.44$.

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Department of Mathematics, University of Washington, Seattle, WA 98195, USA
E-mail address: (Jun Zhang) zhangjun19@gmail.com